Solution 4

1. Suppose X is a Banach space.

Let $Y = \{ f \in X : f(0) = 0 \}$.

Claim: Y is Banach space. wrt the sup norm.

Pf: Let $\{f_n\}$ be a cauchy sequence in Y,

then there exists $f \in X$, (f_n) converges to f uniformly on $(-1, 1)$.

Hence, $0 = \lim_{n\to\infty} f_n(0) = f(0)$, $f \in Y$.

Let $T: X \to Y$ defined by

$$T(f)(x) = \int_0^x f(t)\,dt ,$$

Claim: T is surjective bounded linear operator.

Pf: $\|Tf\|_\infty = \sup_{x \in (-1,1)} |Tf(x)| \leq \sup_{x \in (-1,1)} |x| \cdot \|f\|_\infty = \|f\|_\infty$, thus $\|T\| \leq 1$.

The linearity follows by the definition.

T is surjective as $T(f') = \int_0^x f'(t)\,dt = f(x)$ for any $x \in Y$.

By open mapping thm, T is an open map from X to Y. Let U be the open unit ball in $C^b(-1,1)$, then 0 is interior point of $T(U)$.

Claim: For any $\varepsilon > 0$, $\{ f \in Y : \|f - 0\| < \varepsilon \} \not\subseteq T(U)$.

If the claim holds, 0 cannot be an interior point of $T(U)$, which is contradiction. Therefore X is not a Banach space.

Proof of the claim:

For any $\varepsilon > 0$, choose $f_\varepsilon(x) = \frac{\varepsilon}{2} \cdot \sin(\frac{2}{\varepsilon} x) \in \{ f \in Y : \|f - 0\| < \varepsilon \}$,

then $f_\varepsilon(x) = T(g_\varepsilon)(x) = \int_0^x \cos(\frac{2}{\varepsilon} t)\,dt$, where $g_\varepsilon(x) = \cos(\frac{2}{\varepsilon} x)$.

However $\|g_\varepsilon\|_\infty = 1$, that is, $g_\varepsilon \notin U$.
This implies $\{f \in Y : \|f - 0\| < \varepsilon\} \not\subset T(U)$.

2. (i) ⇒ (ii)

Suppose $(\|T_n x\|)$ is bounded, $\forall x \in X$,
in other words, there exists a const $M_x > 0$, st
$$\sup_{n \geq 1} \|T_n x\| \leq M_x, \quad \forall x \in X.$$
Let $f \in Y^*$, then $|f(T_n x)| \leq \|f\| \cdot \|T_n x\| \leq \|f\| \cdot M_x$

(ii) ⇒ (i)

Suppose $(f(T_n x))$ is bounded, $\forall f \in Y^*$.
Consider the mapping $Q : Y \to Y^{**}$ defined as
$$Q(y)(f) = f(y), \quad \forall y \in Y, \ f \in Y^*.$$
Let $y_n = T_n x$, then we have
$$Q(y_n)(f) = f(y_n) = f(T_n x).$$
Hence, $(Q(y_n)(f))$ is a bounded sequence.
Since Y^* is Banach space, by Unif Boundedness Thm,
$(Q(y_n))$ is a bounded in Y^{**}.
Since Q is an isometry, $\|Q(y_n)\| = \|y_n\| = \|T_n x\|$, which induces that $(\|T_n x\|)$ is bounded.

(i) ⇒ (iii) It follows by uniform boundeness Thm.

(iii) ⇒ (i) Suppose $\|T_n\| \leq M$, $\forall n \in \mathbb{N}$.
Then $\|T_n x\| \leq \|T_n\| \cdot \|x\| \leq M \|x\|$, $\forall x \in X$.
$(\|T_n x\|)_n$ is bounded for any $x \in X$.

3. Let $T: X \to Y$ be closed map.

Suppose a sequence $(y_n)_n$ in Y satisfies

$\lim_{n\to\infty} \| (y_n, T^{-1}y_n) - (y, x) \| = 0$, for some $y \in Y$, $x \in X$.

Let $x_n = T^{-1}y_n$, then we have

$$\lim_{n\to\infty} \| (Tx_n, x_n) - (y, x) \| = 0.$$

By closedness of T, we have $T(x) = y$, which implies

$x = T^{-1}y$.

Therefore, T^{-1} is closed.